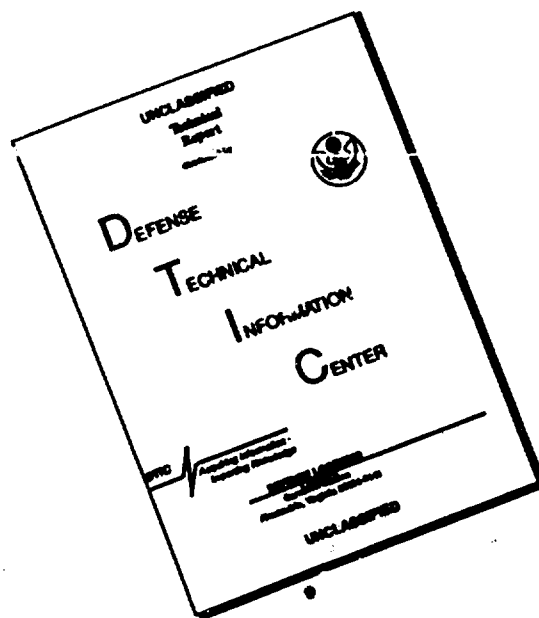# DISCLAIMER NOTICE

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FURTHER RESULTS ON THE ERRORS
IN THE VARIABLES PROBLEM[*]

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I. INTRODUCTION

In surveying the literature on the errors-in-the-variables (EV) model, we are struck by the omission of two important aspects of the model. It is a well known fact that measurement error in one explanatory variable yields an asymptotically downward biased OLS estimator of the coefficient. It is equally well known that measurement error in the dependent variable leads to a reduction in the power of the usual t-statistic in testing the null hypothesis of no relationship between the variables, even though the point esti- mates yielded are unbiased. However, the effects on the power of the t-test when the explanatory variables in an OLS regression contain measurement errors are not generally known. Moreover, the case in which there is measurement error in more than one explanatory variable has not been adequately treated.

In this paper we are concerned with the effects of measurement errors in one or more explanatory variables upon the asymptotic bias of the coefficients and the t-tests. We first look at a simple regression model with one explanatory variable. We do this because the reader will find it somewhat easier to follow the logic in the case of a single explanatory variable. Next we consider a model with two explanatory variables. Finally, we are able to expand to the n-variable model by way of partitioning.

In summary, we find that if one variable is subject to measurement error, its coefficient and t-statistic are asymptotically biased toward zero, even though the sign of the asymptotic bias of its standard error cannot be determined. If more than one explanatory variable is subject to measurement error, the sign of the bias in all coefficients and t-statistics cannot be determined a priori.

II. THE SIMPLE MODEL

Consider the following simple regression model:

(2.1) $y_i = \beta x_i + \epsilon_i$,

where y_i = vector of observations on the dependent variable,

$\quad\quad x_i$ = vector of observations on the explanatory variable,

$\quad\quad \epsilon_i$ = vector of disturbance terms, and

$\quad\quad \epsilon_i \sim NID (0, \sigma_\epsilon^2)$.

However, suppose that the x_i's are subject to measurement error such that the measured independent variable $\bar{x}_i$ is generated by

(2.2) $\bar{x}_i = x_i + u_i$

where $u_i \sim NID (0, \sigma_u^2)$ [1]/ and $E(x_i u_j) = 0$ for all i, j.

Therefore, the following model is estimated:

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[1] By assuming that measurement errors are distributed normally, we can show that although the power of the t test is reduced, the t test is still an appropriate test. If the u_i's are not distributed normally, then we cannot even say that the t test is appropriate. However, the remainder of our results (i.e., except for the applicability of the t test) still hold if we simply assume that $E(u_i) = 0$ and $E(u_i u_j) = 0$ for $i \neq j$ and $= \sigma_u^2$ for $i = j$.

$$(2.3) \qquad y_i = b \, x_i + v_i \; ,$$

where $v_i = \varepsilon_i - b \, u_i$.

It is a well known result that the least squares estimator $\hat{b}$ is an asymptotically biased (toward zero) estimator of β. This is shown below:

$$\hat{b} = \frac{\sum\limits_i \bar{x}_i \, y_i}{\sum\limits_i \bar{x}_i^2} \; {}^{1/}$$

$$= \frac{\sum (x+u)(\beta x + \varepsilon)}{\sum (x+u)^2}$$

$$= \frac{\beta \sum x^2 + \sum x \, \varepsilon + \beta \sum xu + \sum u \, \varepsilon}{\sum x^2 + 2 \sum x \, u + \sum u^2}$$

Then, if

$$\operatorname*{plim}_{T\to\infty} (\tfrac{1}{T} \sum x \, \varepsilon) = \operatorname*{plim}_{T\to\infty} (\tfrac{1}{T} \sum u \, \varepsilon) = \operatorname*{plim}_{T\to\infty} (\tfrac{1}{T} \sum x \, u) = 0 \; {}^{2/}$$

$$\operatorname{plim} (\hat{b}) = \frac{\operatorname{plim} (\tfrac{1}{T} \beta \sum x^2)}{\operatorname{plim} (\tfrac{1}{T} \sum x^2 + \tfrac{1}{T} \sum u^2)}$$

$$= \frac{\beta \, \sigma_x^2}{\sigma_x^2 + \sigma_u^2}$$

$$(2.4) \qquad\qquad = \beta \left(\frac{1}{1+\lambda} \right) \; ,$$

where $\operatorname{plim} (\tfrac{1}{T} \sum x^2) = \sigma_x^2$ and $\operatorname{plim} (\tfrac{1}{T} \sum u^2) = \sigma_u^2 = \lambda \, \sigma_x^2$; $\lambda > 0$.

Therefore, the estimator $\hat{b}$ is asymptotically biased toward zero. In fact, as $\lambda \to \infty$, $1/(1 + \lambda) \to 0$ and $\hat{b} \to 0$.

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[1] For notational simplicity, the "i" subscripts are hereafter deleted.

[2] Again, for notational simplicity, "$\operatorname*{plim}_{T\to\infty}$" will be written "plim."

Next, consider the estimator of residual variance in the EV model (2.2)-(2.3):

$$(2.5) \qquad \hat{\sigma}_v^2 = \frac{1}{T} \Sigma \ (y - \hat{b} \ \bar{x})^2$$

$$= \frac{1}{T} \Sigma \ y^2 - \frac{1}{T} \ 2 \ \hat{b} \ \Sigma \ (x + u)(\beta \ x + \epsilon) + \frac{1}{T} \ \hat{b}^2 \ \Sigma (x + u)^2.$$

Then, through manipulation of (2.5) the probability limit of $\hat{\sigma}_v^2$ may be given

$$\text{plim } \hat{\sigma}_v^2 = \sigma_y^2 - \beta^2 \ (\frac{1}{1 + \lambda}) \ \sigma_x^2$$

Since $\sigma_\epsilon^2 = \sigma_y^2 - \beta^2 \ \sigma_x^2$, the estimator $\hat{\sigma}_v^2$ is an asymptotically biased (upwards) estimator of σ_ϵ^2. The asymptotic bias is given

$$\text{plim } (\hat{\sigma}_v^2 - \sigma_\epsilon^2) = (\frac{\lambda}{1 + \lambda}) \ \beta^2 \ \sigma_x^2$$

Third, consider the EV model estimator of the variance of the $\hat{b}$ coefficient. The EV model estimator is given by

$$\hat{\sigma}_b^2 = \frac{\hat{\sigma}_v^2}{\sigma_{\bar{x}}^2}$$

and its probability limit by

$$(2.6) \qquad \text{plim } \hat{\sigma}_b^2 = \frac{\text{plim } \hat{\sigma}_v^2}{\text{plim } \sigma_{\bar{x}}^2}$$

$$= \frac{\sigma_\epsilon^2 + (\frac{\lambda}{1 + \lambda}) \ \beta^2 \ \sigma_x^2}{(1 + \lambda) \ \sigma_x^2} \ ,$$

while $\sigma_\beta^2 = \dfrac{\sigma_\epsilon^2}{\sigma_x^2}$.

Since both the numerator and denominator in (2.6) are biased upward, the direction of bias in $\hat{\sigma}_b$ cannot, in general, be determined, for:

$$(2.7) \qquad \text{plim } (\hat{\sigma}_b^2 - \sigma_\beta^2) = \frac{\lambda}{1+\lambda} \left(\frac{\beta^2}{\lambda+1} - \frac{\sigma_\epsilon^2}{\sigma_x^2}\right)$$

$$> 0 \quad \text{if} \quad \beta^2 > (\lambda + 1) \frac{\sigma_\epsilon^2}{\sigma_x^2}$$

$$< 0 \quad \text{if} \quad \beta^2 < (\lambda + 1) \frac{\sigma_\epsilon^2}{\sigma_x^2}$$

Thus, without prior knowledge on the relation between β^2 and $\sigma_\epsilon^2/\sigma_x^2$, the sign of the asymptotic bias of $\hat{\sigma}_b^2$ cannot be determined.

Therefore, one might conclude that since the "t" ratio is the ratio of the estimated coefficient to its standard error, the direction of bias for the t ratio cannot be determined _a priori_ in the EV model. However, as shown below, this is not the case. For, the EV model the t ratio is given by:

$$(2.8a) \qquad \hat{t}_b = \frac{\hat{b}}{\hat{\sigma}_b} = \frac{\hat{b}}{(\sigma_v/\sigma_x)}$$

or

$$(2.8b) \qquad \hat{t}_b = \frac{\hat{b}\,\sigma_x}{\hat{\sigma}_v}$$

$$\text{and plim } (\hat{t}_b) = \frac{\beta\left(\frac{1}{1+\lambda}\right)(1+\lambda)^{1/2}\sigma_x}{\left(\sigma_\epsilon^2 + \beta^2\left(\frac{\lambda}{1+\lambda}\right)\sigma_x^2\right)^{1/2}}$$

$$(2.9) \qquad = \frac{\beta(1+\lambda)^{-1/2}\sigma_x}{(1+\mu)^{1/2}\sigma_\epsilon} < \frac{\beta\,\sigma_x}{\sigma_\epsilon} = t_\beta$$

$$\text{where} \qquad \mu = \beta^2\left(\frac{\lambda}{1+\lambda}\right)\frac{\sigma_x^2}{\sigma_\epsilon^2}$$

That is, the estimator $\hat{t}_b$ is asymptotically biased downward, for the numerator on the left hand side (l.h.s.) of (2.9) is biased downward, while the denominator on the l.h.s. of (2.9) is biased upward. Note that since u is normally distributed (see the footnote on p. 2), the t test is still an appropriate test of $H_o: \beta = 0$, because under H_o $\hat{b}$ is NID $(0, \sigma_\varepsilon^2 + b^2 \sigma_u^2)$ and $\hat{\sigma}_v^2$ is chi-square with n-2 degrees of freedom.

III. THE TWO EXPLANATORY VARIABLE MODEL

Consider the following two explanatory variable regression model:

(3.1) $y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \varepsilon_i$,

where x_{1i} and x_{2i} are the two vectors of observation on the two explanatory variables. However, suppose that x_{1i} is subject to measurement error, such that the measured explanatory variable $\bar{x}_{1i}$ is generated by

(3.2) $\bar{x}_{1i} = x_{1i} + u_i$,

where u_i · NID $(0, \sigma_u^2)$ and $E(x_{ij} u_k) = 0$ for all i, j, k. Therefore, the following model is estimated:

(3.3) $y_i = b_1 \bar{x}_{1i} + b_2 x_{2i} + v_i$

where $v_i = \varepsilon_i - b_1 u_i$.

Like the single variable case, it is a well known result that, if x_1 and x_2 are orthogonal to one another, the estimator $\hat{b}_1$ is an asymptotically biased estimator (toward zero) of β_1 and the estimator b_2 is a consistent estimator of β_2. However, results when x_1 and x_2 are not orthogonal are not so well known.

To consider the two variable model, we suppose that the following probability limits exist and are defined as:

$$\text{plim} \frac{1}{T} \Sigma \, x_1^{\,2} = \sigma_{11}$$

$$\text{plim} \frac{1}{T} \Sigma \, u^2 = \sigma_{uu} = \lambda \, \sigma_{11}$$

$$\text{plim} \frac{1}{T} \Sigma \, \bar{x}_1^{\,2} = \sigma_{11} + \sigma_{uu} = (1+\lambda) \, \sigma_{11}$$

$$\text{plim} \frac{1}{T} \Sigma \, x_2^{\,2} = \sigma_{22}$$

$$\text{plim} \frac{1}{T} \Sigma \, x_1 \, x_2 = \sigma_{12}$$

(3.4)

$$\text{plim} \frac{1}{T} \Sigma \, y^2 = \sigma_{yy}$$

$$\text{plim} \frac{1}{T} \Sigma \, \epsilon^2 = \sigma_{\epsilon\epsilon} = \sigma_\epsilon^{\,2}$$

$$\text{plim} \frac{1}{T} \Sigma \, x_1 \, u = 0$$

$$\text{plim} \frac{1}{T} \Sigma \, x_2 \, u = 0$$

$$\text{plim} \frac{1}{T} \Sigma \, \epsilon \, u = 0.$$

Then, the estimator $\hat{b} = \begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix}$ is defined:

$$\hat{b} = (\bar{X}' \, \bar{X})^{-1} \, \bar{X}' \, y$$

$$= (\bar{X}' \, \bar{X})^{-1} \, \bar{X}' \, X \, \beta + (\bar{X}' \, \bar{X})^{-1} \, \bar{X}' \, \epsilon$$

where $X = (x_{1i}, \, x_{2i})$ and $\bar{X} = (\bar{x}_{1i}, \, x_{2i})$.

The probability limit of $\hat{b}$ is given

$$\text{plim} \, \hat{b} = \text{plim} \, [(\bar{X}' \, \bar{X})^{-1} \, \bar{X} \, X \, \beta] + \text{plim} \, [(\bar{X}' \, \bar{X})^{-1} \, \bar{X}' \, \epsilon]$$

$$= \text{plim} \, [(T^{-1} \, \bar{X}' \, \bar{X})^{-1} \, (T^{-1} \, \bar{X} \, X) \, \beta]$$

$$= \Sigma_{\bar{X} \, \bar{X}}^{-1} \cdot \Sigma_{XX} \cdot \beta$$

where $\quad \Sigma_{\overline{X}\,\overline{X}} = \begin{pmatrix} (1+\lambda)\ \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$ and $\quad \Sigma_{XX} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}'.$

Therefore,

$$\text{plim}\ \hat{b} = \begin{pmatrix} (1+\lambda)\ \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}^{-1} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

$$= \left\{ \begin{matrix} \dfrac{\sigma_{11}\ \sigma_{22} - \sigma_{12}^{\,2}}{(1+\lambda)\ \sigma_{11}\ \sigma_{22} - \sigma_{12}^{\,2}} & 0 \\[3mm] \dfrac{\lambda\ \sigma_{11}\ \sigma_{12}}{(1+\lambda)\ \sigma_{11}\ \sigma_{22} - \sigma_{12}^{\,2}} & 1 \end{matrix} \right\} \left\{ \begin{matrix} \beta_1 \\[3mm] \beta_2 \end{matrix} \right\}$$

$$= \begin{pmatrix} \alpha & 0 \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

(3.5) $$= \begin{pmatrix} \alpha\ \beta_1 \\ \beta_2 + \gamma\ \beta_1 \end{pmatrix},$$

where

$$\alpha = \frac{\sigma_{11}\ \sigma_{22} - \sigma_{12}^{\,2}}{(1+\lambda)\ \sigma_{11}\ \sigma_{22} - \sigma_{12}^{\,2}}\ ,\ 0 \le \alpha \le 1\ ,$$

and

$$\gamma = \frac{\lambda\ \sigma_{11}\ \sigma_{12}}{(1+\lambda)\ \sigma_{11}\ \sigma_{22} - \sigma_{12}^{\,2}}\ .$$

Therefore, the estimator $\hat{b}_1$ is an asymptotically biased estimator (toward zero) of β_1. However, although $\hat{b}_2$ is an asymptotically biased estimator of β_2, the sign of the bias cannot be determined _a priori_: $\hat{b}_2$ is biased upward if β_1 and σ_{12} have the same sign and downward if β_1 and σ_{12} are opposite in sign. Note that if x_1 and x_2 are orthogonal (i.e., $\sigma_{12} = 0$) $\hat{b}_1$ is still biased toward zero but $\hat{b}_2$ is an unbiased estimator of β_2 since $\gamma = 0$.

Next, consider the estimator of residual variance, $\hat{\sigma}_v^2$, in the two variable EV model:

$$\hat{\sigma}_v^2 = (y - \overline{X} \hat{b})' \, (y - \overline{X} \hat{b})$$

$$= y' \, y - 2 \hat{b}' \, \overline{X}' \, (X \beta + \varepsilon) + \hat{b}' \, \overline{X}' \, \overline{X} \, \hat{b} \; .$$

The probability limit, as $T \to \infty$, of $\hat{\sigma}_v^2$ is given by:

$$(3.6) \qquad \text{plim } \hat{\sigma}_v^2 = \sigma_{yy} - 2 \, b^{*'} \Sigma_{XX} \, \beta + b^{*'} (\Sigma_{\overline{XX}}) \, b^* ,$$

where $b^* = \text{plim } \hat{b}$. Equation (3.6) may be rewritten

$$\text{plim } \hat{\sigma}_v^2 = \sigma_{yy} - 2 (\alpha \, \beta_1, \; \beta_2 + \gamma \, \beta_1) \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

$$+ (\alpha \, \beta_1, \; \beta_2 + \gamma \, \beta_1) \begin{pmatrix} (1+\lambda) \, \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \alpha \, \beta_1 \\ \beta_2 + \gamma \, \beta_1 \end{pmatrix}$$

$$= \sigma_{yy} + [(1+\lambda) \, \alpha^2 - 2\alpha] \, \beta_1^2 \, \sigma_{11} - 2 \, \beta_1 \, \beta_2 \, \sigma_{12} + 2\gamma(\alpha-1) \, \beta_1^2 \, \sigma_{12}$$

$$+ \gamma^2 \, \beta_1^2 \, \sigma_{22} - \beta_2^2 \, \sigma_{22}$$

$$= \sigma_v^{*2}$$

Since $\sigma_\varepsilon^2 = \sigma_{yy} - \beta_1^2 \, \sigma_{11} - 2 \, \beta_1 \, \beta_2 \, \sigma_{12} - \beta_2^2 \, \sigma_{22}$, the asymptotic bias of $\hat{\sigma}_v^2$ is given by:

$$\text{plim } (\hat{\sigma}_v^2 - \sigma_\varepsilon^2) = [(\alpha-1)^2 + \lambda \, \alpha^2] \, \beta_1^2 \, \sigma_{11} + 2\gamma(\alpha-1) \, \beta_1^2 \, \sigma_{12} + \gamma^2 \, \beta_1^2 \, \sigma_{22}$$

$$\geq 0 \quad [1]$$

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[1] This may be proved by substituting in for γ and α.

That is, $\hat{\sigma}_v^2$ is an asymptotically biased (upward) estimator of σ_ϵ^2.

The estimator of the variance of the coefficients in the two variable EV model is given by the diagonal elements of the matrix

$$\hat{\sigma}_b^2 = \hat{\sigma}_v^2 \; (\overline{X}' \; \overline{X})^{-1} \; .$$

The probability limit of $\hat{\sigma}_b^2$ may be written

$$(3.7) \qquad \text{plim} \; \hat{\sigma}_b^2 = \sigma_v^{*2} \begin{pmatrix} \sigma_{22}/ \overline{D} \\ (1+\lambda) \; \sigma_{11}/\overline{D} \end{pmatrix},$$

while

$$\sigma_\beta^2 = \sigma_\epsilon^2 \begin{pmatrix} \sigma_{22}/D \\ \sigma_{11}/D \end{pmatrix},$$

where $D = \det (\Sigma_{XX})$ and $\overline{D} = \det (\Sigma_{\overline{XX}})$.

Just as with the one variable EV model, the sign of the asymptotic bias of the estimator of the variance of the coefficients in the two variable EV model cannot be determined a priori, for both the numerator and the denominator for each of the elements in (3.7) is biased upward.

However, in spite of the fact that the sign of the bias of the standard errors of the coefficients cannot be determined, we are able to determine the bias in one of the t-statistics. Whereas the true t-statistics are given

$$t_{\beta_1} = \frac{\beta_1}{\sigma_{\beta_1}} = \frac{\beta_1 \; \sqrt{D}}{\sigma_\epsilon \cdot \sigma_2}$$

and

$$t_{\beta_2} = \frac{\beta_2}{\sigma_{\beta_2}} = \frac{\beta_2 \; \sqrt{D}}{\sigma_\epsilon \cdot \sigma_1} \; , \qquad \text{where } \sigma_i = \sigma_{ii}^{1/2}$$

The t-statistics in the EV model are

$$\hat{t}_{b_1} = \frac{\hat{b}_1}{\hat{\sigma}_{b_1}}$$

and

$$\hat{t}_{b_2} = \frac{\hat{b}_2}{\hat{\sigma}_{b_2}} .$$

Then,

$$\text{plim} (\hat{t}_{b_1}) = \frac{\alpha \ \beta_1}{\sigma_v^* \ \sqrt{\frac{\sigma_{22}}{\overline{D}}}} = \frac{\alpha \sqrt{\overline{D}} \ \beta_1}{\sigma_v^* \ \sigma_2} .$$

Since $\alpha = \frac{D}{\overline{D}}$, $\text{plim} (\hat{t}_{b_1})$ may be rewritten

$$(3.8) \qquad \text{plim} (\hat{t}_{b_1}) = \frac{\frac{D}{\overline{D}} \ \sqrt{\overline{D}} \ \beta_1}{\sigma_v^*} = \frac{\frac{\sqrt{D}}{\sqrt{\overline{D}}} \cdot \sqrt{D} \ \beta_1}{\sigma_v^* \ \sigma_2}$$

Therefore,

$$(3.9) \qquad |\text{plim} (\hat{t}_{b_1})| \ \leq \ | \ t_\beta \ |$$

since $\left| \frac{\sqrt{D}}{\sqrt{\overline{D}}} \right| \leq 1$ and $\sigma_v^* \geq \sigma_\epsilon$.

That is, the t-statistics for b_1 in the two variable EV model is asymptotically biased toward zero.

On the other hand,

$$(3.10) \qquad \text{plim} (\hat{t}_{b_2}) = \frac{\beta_2 + \gamma \ \beta_1}{\sigma_v^* \ \sqrt{(1+\lambda) \ \sigma_{11}/\overline{D}}} .$$

which suggests that the sign of the bias of $\hat{t}_{b_2}$ is not known <u>a priori</u>, for b_2 may be biased upward (if β_1 and σ_{12} are the same in sign) or biased downward (if β_1 and σ_{12} are opposite in sign).

Thus, (3.9) and (3.10) indicate that although the bias in the t-test of the coefficient of the variable which <u>does not</u> contain measurement error is not known, the t-test of the coefficient of the

variable which <u>does</u> contain measurement error is biased toward zero. Again, as in the one variable EV model, the t-test is an appropriate test of the null hypothesis: H_o: $\beta_1 = 0$ because under H_o, $\hat{b}_1$ is NID(0, $\sigma_{b_1}^2$) and $\hat{\sigma}_v^2$ is chi-square with n-3 degrees of freedom.

IV. THE n-VARIABLE EV MODEL

Consider the following multiple regression model

$$(4.1) \qquad y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \dots + \beta_n x_{ni} + \epsilon_i \text{ ,}$$

where the x_{ji}'s are the explanatory variables. However, suppose that x_{1i} is subject to measurement error, such that the measured explanatory variable $\bar{x}_{1i}$ is generated by

$$(4.2) \qquad \bar{x}_{1i} = x_{1i} + u_i \text{ ,}$$

where $u_i \sim$ NID (0, σ_u^2).and $E(x_{ij} u_k) = 0$ for all i, j, k.

Therefore, the following model is estimated

$$(4.3) \qquad y_i = b_1 \bar{x}_{1i} + b_2 x_{2i} + b_3 x_{3i} + \dots + b_n x_{ni} + v_i \text{ ,}$$

where $v_i = \epsilon_i - b_1 u_i$.

Again, as in the one variable model, it is a well known result that $\hat{b}_1$ is an asymptotically biased estimator of β_1 and that $\hat{b}_i$, i = 2, n, are unbiased estimators of β_i, i = 2, n, if x_1 is orthogonal to every other x_i. However, to our knowledge, results have not been established in the use in which x_1 is not orthogonal to the remaining variables.

This n-variable model may be analyzed by partitioning the $\bar{X}$ matrix into two parts: $\bar{x}_1$ and every other x. Thus, $\bar{X} = (\bar{x}_1, X_2)$ X_2 is a t x(n-1) matrix of t observations on the remaining n-1 variables.

Now treating X_2 as x_2 in the two-variable model, we can obtain
the same results in the n-variable model as in the two variable model:
namely, (1) $\hat{b}_1$ is biased toward zero, (2) the sign of the biases in
$\hat{b}_2, \ldots \hat{b}_n$ is indeterminant, (3) the t-ratio for $\hat{b}_1$ is biased toward
zero, and (4) the direction of the biases in the t-ratios for $\hat{b}_2, \ldots \hat{b}_n$
is indeterminant.

Notice, however, that the n-variable model considered here is
not the most general case, for we have considered only one variable
with measurement error. In fact, the results derived here will not
in general hold when more than one of the variables contain measure-
ment error. This can be shown by returning to the two variable
model. Consider the two variable model in which both explanatory
variables contain measurement error:

(4.4) $\bar{x}_{1i} = x_{1i} + u_{1i}$

$\bar{x}_{2i} = x_{2i} + u_{2i}$

where $u_{ji} \sim \text{NID} (0, \lambda_j \sigma_{jj})$.

Then,

$$\text{plim } (\hat{b}) = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \gamma_1 & \alpha_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \beta_1 + \gamma_2 \beta_2 \\ \alpha_2 \beta_2 + \gamma_1 \beta_1 \end{pmatrix}$$

where $\alpha_1 = \dfrac{(1+\lambda_2) \sigma_{11} \sigma_{22} - \sigma_{12}^2}{(1+\lambda_1)(1+\lambda_2) \sigma_{11} \sigma_{22} - \sigma_{12}^2}$,

$\alpha_2 = \dfrac{(1+\lambda_1) \sigma_{11} \sigma_{22} - \sigma_{12}^2}{(1+\lambda_1)(1+\lambda_2) \sigma_{11} \sigma_{22} - \sigma_{12}^2}$,

$\gamma_1 = \dfrac{\lambda_1 \sigma_{11} \sigma_{12}}{(1+\lambda_1)(1+\lambda_2) \sigma_{11} \sigma_{22} - \sigma_{12}^2}$,

and $\quad \gamma_2 = \dfrac{\lambda_2 \; \sigma_{22} \; \sigma_{12}}{(1+\lambda_1)(1+\lambda_2) \; \sigma_{11} \; \sigma_{22} - \sigma_{12}^{\;2}}$

Therefore, unlike the case of measurement error in only one explanatory variable, the direction of bias in the coefficients can no longer be determined a priori. As a result, the previous conclusions regarding the t-statistic are also no longer valid.